

Shifted Poisson and Batalin-Vilkovisky structures on the derived variety of complexes

Slava Pimenov

November 4, 2015

0 Introduction.

Let V^\bullet be a graded vector space of finite total dimension over field k . The derived variety of complexes ([KP]) is a dg-scheme $\mathrm{RCom}(V^\bullet)$ characterized by the following property: for a commutative differential graded algebra A over k the set of maps $\mathrm{Hom}(\mathrm{Spec} A, \mathrm{RCom}(V^\bullet))$ is the set of differentials in the total complex of $A \otimes V^\bullet$ “going in the direction of V^\bullet ”, making it a twisted complex.

In this paper we will describe a 1-shifted Poisson structure on the quotient of $\mathrm{RCom}(V^\bullet)$ by the infinitesimal action of a subgroup of automorphisms of V^\bullet with superdeterminant 1. Moreover, this Poisson structure can be upgraded (up to a quasi-isomorphism) to a structure of Batalin-Vilkovisky manifold. Since the ring of functions on $\mathrm{RCom}(V^\bullet)$ can be explicitly expressed as a cochain complex of certain graded Lie algebra associated to V^\bullet we look at this Lie algebra for a source of such structures.

0.1 If G is a classical Poisson-Lie group, the Schouten-Nijenhuis bracket on the polyvector fields translates to a bracket on the differential forms $\Omega^\bullet G$. Together with the de Rham differential and product it equips Ω^\bullet with a differential Gerstenhaber algebra structure ([Kos]).

We mimic this construction in the case of n -shifted graded Lie bialgebra \mathfrak{g} to obtain an $(n+1)$ -shifted Poisson structure on the cochain complex $C^\bullet(\mathfrak{g}, k)$.

0.2 In the second section we apply this result in the case of derived variety of complexes and using the standard bialgebra structure on $\mathfrak{gl}(m, n)$ obtain a 1-Poisson structure on the infinitesimal quotient $\mathrm{RCom}(V^\bullet)/\mathfrak{l}_1$, where \mathfrak{l}_1 is a subalgebra of $\mathfrak{gl}(V^\bullet)$ of supertrace 0 elements.

We also define a derived variety of 1-periodic complexes, that possesses a non-shifted Poisson structure, extending Kirillov-Kostant structure on \mathfrak{gl}_n^* .

0.3 Although the cochain complex of a graded Lie bialgebra does not necessarily possess a Batalin-Vilkovisky operator, in some cases it is quasi-isomorphic as a differential Gerstenhaber algebra to a differential BV-algebra. In the last section we give a sufficient condition for existence of such quasi-isomorphism, which applies to the case of the quotient $\mathrm{RCom}(V^\bullet)/\mathfrak{l}_1$.

The author would like to thank Kavli IPMU for providing wonderful working conditions, and also Mikhail Kapranov and Alexander Voronov for their invaluable advice in preparation of this paper.

1 Shifted Lie bialgebras.

Let $V^\bullet = \bigoplus V^i$ be a graded vector space over field k , such that each V^i is of finite dimension.

Definition 1.1 *An n -shifted Lie bialgebra is a graded Lie algebra $(V^\bullet, [-, -])$, with a cobracket $\varphi: V^\bullet[-n] \rightarrow V^\bullet[-n] \otimes V^\bullet[-n]$, such that the dual map φ^* is a graded Lie algebra structure on $(V^\bullet[-n])^*$, and the composition*

$$V \xrightarrow{\varphi[n]} (V[-n] \otimes V[-n])[n] \xrightarrow{\simeq} (V \otimes V)[-n]$$

is a 1-cocycle with coefficients in $(V \otimes V)[-n]$, i.e., it satisfies the following condition:

$$\varphi([x, y]) = \text{ad}_x \varphi(y) - (-1)^{\bar{x}\bar{y}} \text{ad}_y \varphi(x),$$

where \bar{x} and \bar{y} are degrees of x and y in V^\bullet .

Abusing notation we denote the cocycle also by φ .

Definition 1.2 *An n -shifted Manin triple is a triple of graded Lie algebras $(\mathfrak{p}, \mathfrak{p}_+, \mathfrak{p}_-)$, with non-degenerate invariant symmetric bilinear form $(-, -): S^2 \mathfrak{p} \rightarrow k[n]$, such that $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ as a graded vector space, \mathfrak{p}_+ and \mathfrak{p}_- are Lie subalgebras of \mathfrak{p} isotropic with respect to the bilinear form.*

We have the following analog of the classical result:

Lemma 1.3 *n -shifted Lie bialgebra structures on \mathfrak{g} are in bijection with n -shifted Manin triples $(\mathfrak{p}, \mathfrak{p}_+, \mathfrak{p}_-)$ with $\mathfrak{p}_+ = \mathfrak{g}$.*

We begin with a Manin triple and will construct the bialgebra structure on $\mathfrak{g} = \mathfrak{p}_+$. Using the bilinear form, we can identify \mathfrak{p}_- with $\mathfrak{p}_+^*[n]$ by sending x to $(x, -)$. Restriction of the Lie bracket to \mathfrak{p}_- defines the cobracket φ^* . Let $\{e_i\}$ be a basis of \mathfrak{g} , and $\{f^i\}$ — the dual basis of \mathfrak{g}^* . We will denote the element of \mathfrak{p}_- corresponding to f^i by $f^i[n]$, so that $(f^i[n], e_j) = \delta_j^i$.

Let $[e_i, e_j] = c_{ij}^k e_k$ and $\varphi^*(f^i[n], f^j[n]) = \gamma_k^{ij} f^k[n]$. Using the invariance of the bilinear form we find that

$$[e_j, f^i[n]] = (-1)^{\bar{e}_i(n+1)} \gamma_j^{ik} e_k - (-1)^{\bar{e}_j(\bar{f}^i - n)} c_{jk}^i f^k[n].$$

Expanding the Jacobi identity we obtain

$$\begin{aligned}
& [[e_i, e_j], f^k[n]] - [e_i, [e_j, f^k[n]]] + (-1)^{\overline{e_i e_j}} [e_j, [e_i, f^k[n]]] = \\
& (-1)^{1+(\overline{f^k}-n)\overline{e_m}} c_{mp}^k c_{ij}^m f^p[n] + (-1)^{\overline{e_k}(n+1)} \gamma_m^{kq} c_{ij}^m e_q + \\
& (-1)^{1+\overline{e_k}(n+1)} \gamma_j^{km} c_{im}^q e_q + (-1)^{\overline{e_j}(\overline{f^k}-n)+\overline{e_m}(n+1)} \gamma_i^{mq} c_{jm}^k e_q + \\
& (-1)^{1+\overline{e_j}(\overline{f^k}-n)+\overline{e_i}(\overline{f^m}-n)} c_{jm}^k c_{ip}^m f^p[n] + \\
& (-1)^{\overline{e_i e_j}+\overline{e_k}(n+1)} \gamma_i^{km} c_{jm}^q e_q + (-1)^{1+\overline{e_i e_j}+\overline{e_i}(\overline{f^k}-n)+\overline{e_m}(n+1)} \gamma_j^{mq} c_{im}^k e_q + \\
& (-1)^{1+\overline{e_i e_j}+\overline{e_i}(\overline{f^k}-n)+\overline{e_j}(\overline{f^m}-n)} c_{im}^k c_{jp}^m f^p[n]
\end{aligned}$$

Combining coefficients of $f^p[n]$ we obtain the Jacobi identity for \mathfrak{g} . Now we want to show that the coefficients of e_q provide the cocycle condition for φ . Using structure constants γ we write

$$\varphi(e_k) = (-1)^{\overline{e_i e_j}+n\overline{e_i}} \gamma_k^{ij} e_i \otimes e_j[-n] \in (\mathfrak{g} \otimes \mathfrak{g})[-n],$$

and the cocycle condition becomes

$$\begin{aligned}
& (-1)^{\overline{e_k e_q}+n\overline{e_k}} \gamma_m^{kq} c_{ij}^m = (-1)^{\overline{e_m e_q}+n\overline{e_m}} \gamma_j^{mq} c_{im}^k + (-1)^{\overline{e_k e_m}+n\overline{e_k}+\overline{e_i e_k}} \gamma_j^{km} c_{im}^q + \\
& (-1)^{1+\overline{e_i e_j}+\overline{e_m e_q}+n\overline{e_m}} \gamma_i^{mq} c_{jm}^k + (-1)^{1+\overline{e_i e_j}+\overline{e_k e_m}+n\overline{e_k}+\overline{e_j e_k}} \gamma_i^{km} c_{jm}^q.
\end{aligned}$$

It remains to compare the signs of the corresponding coefficients.

The proof of the lemma in the other direction is similar.

Proposition 1.4 *For an n -shifted Lie bialgebra \mathfrak{g} the dual $\mathfrak{g}' = \mathfrak{g}^*[n]$ is also an n -shifted Lie bialgebra.*

Immediately follows by interchanging roles of \mathfrak{p}_+ and \mathfrak{p}_- in the Manin triple associated to \mathfrak{g} .

Proposition 1.5 *Let $(\mathfrak{g}, [-, -], \varphi)$ be an n -shifted Lie bialgebra, then the cochain complex with trivial coefficients $C^\bullet(\mathfrak{g}) = C^\bullet(\mathfrak{g}, k) = S^\bullet(\mathfrak{g}^*[-1])$ is a differential $(n+1)$ -shifted Poisson algebra.*

We will denote by $|a|$ the degree of a cochain a in $C^\bullet(\mathfrak{g})$, so that for any $x \in \mathfrak{g}^*$, considered as a cochain we have $|x| = \bar{x} + 1$. Define the Poisson bracket on $\mathfrak{g}^*[-1]$ as $\langle x[-1], y[-1] \rangle = \varphi^*(x[n], y[n]) \in (\mathfrak{g}^*)^{\bar{x}+\bar{y}-n} \subset C^{|x|+|y|-n-1}(\mathfrak{g})$, and extend to the entire cochain complex using the derivation relation $\langle x, yz \rangle = \langle x, y \rangle z + (-1)^{(|x|-n-1)|y|} y \langle x, z \rangle$. The commutation and Jacobi identities for the Poisson bracket are equivalent to those of the cobracket φ^* .

It remains to show the compatibility of the derivation d of the cochain complex and the Poisson bracket:

$$d \langle x, y \rangle = \langle dx, y \rangle + (-1)^{|x|-n-1} \langle x, dy \rangle. \quad (1.5.1)$$

Using notations from the previous lemma we have

$$\begin{aligned}\langle f^i[-1], f^i[-1] \rangle &= \gamma_k^{ij} f^k[-1], \\ d(f^k[-1]) &= (-1)^{\overline{e_p e_q} + \overline{e_p}} c_{pq}^k f^p[-1] f^q[-1],\end{aligned}$$

and it is easy to see that the relation (1.5.1) is equivalent to the cocycle condition for the Lie bialgebra.

Corollary 1.6 *Let $(\mathfrak{g}, [-, -], \varphi)$ be an n -shifted Lie bialgebra, then the cohomology ring $H^\bullet(\mathfrak{g}, k)$ is an $(n+1)$ -shifted Poisson algebra.*

Example 1.7 Let \mathfrak{g} be a graded Lie algebra, it has always the trivial n -shifted Lie bialgebra structure given by the zero cobracket. The dual Lie bialgebra \mathfrak{g}' is an abelian Lie algebra with an n -shifted cobracket. Its cohomology $H^\bullet(\mathfrak{g}', k) = C^\bullet(\mathfrak{g}') = S^\bullet(\mathfrak{g}[-n-1])$ is an $(n+1)$ -shifted Poisson algebra. For $n = 0$ this is the usual Gerstenhaber algebra structure on the $\Lambda^\bullet \mathfrak{g}$ (in fact a Batalin-Vilkovisky algebra).

Example 1.8 Now let \mathfrak{g} be the graded Lie algebra $\mathfrak{gl}(m, n)$, and fix a Borel subalgebra \mathfrak{b}_+ in \mathfrak{g} . Denote by \mathfrak{b}_- the opposite Borel subalgebra. We use Manin triples to define a (non-shifted) Lie bialgebra structure on it. \mathfrak{g} is spanned by elementary matrices e_{ij} with 1 in position (i, j) and 0 otherwise, and diagonal matrices a_i with 1 in position (i, i) . Consider the supertrace bilinear form $(a, b) = \text{tr}(ab)$ on \mathfrak{g} . In this basis it is given by $(e_{ij}, e_{ji}) = (-1)^{\alpha_i}$, $(a_i, a_i) = (-1)^{\alpha_i}$ and 0 otherwise. Here α_i is the degree of the i -th basis vector in the defining representation of \mathfrak{g} , i.e. $\alpha_i = 0$ for $1 \leq i \leq m$ and $\alpha_i = 1$ for $m < i \leq m+n$. This is a non-degenerate symmetric \mathfrak{g} -invariant bilinear form.

Consider the triple consisting of $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{g}$ as a graded Lie algebra, \mathfrak{p}_+ — the image of the diagonal map from \mathfrak{g} to \mathfrak{p} , and $\mathfrak{p}_- = \{(x, y) \in \mathfrak{b}_+ \oplus \mathfrak{b}_- \mid p(x) + p(y) = 0\}$, where $p: \mathfrak{b}_\pm \rightarrow \mathfrak{h}$ are projections of the Borel subalgebras on the Cartan subalgebra. Extend the bilinear form from \mathfrak{g} to \mathfrak{p} by $((x_1, y_1), (x_2, y_2)) = (x_1, x_2) - (y_1, y_2)$. This turns the triple $(\mathfrak{p}, \mathfrak{p}_+, \mathfrak{p}_-)$ into a Manin triple.

If \mathfrak{q} is parabolic subalgebra of \mathfrak{g} , containing \mathfrak{b}_+ , then the orthogonal $\mathfrak{q}^\perp \in \mathfrak{g}^*$ is identified via the form on \mathfrak{p} with a nilpotent ideal in \mathfrak{b}_- . Therefore the cobracket φ restricts to \mathfrak{q} and obviously satisfies the cocycle condition.

Example 1.9 Let $\mathfrak{b} \subset \mathfrak{gl}_n$ is a Borel subalgebra with the restriction of the standard bialgebra structure on \mathfrak{gl}_n . In this case the cohomology groups $H^\bullet(\mathfrak{b}, k) = \Lambda^\bullet \mathfrak{h}^*$, where \mathfrak{h} is the Cartan subalgebra of \mathfrak{gl}_n , and the induced Poisson bracket is trivial.

However this is not the case for the graded Lie algebras. For example, consider a Borel subalgebra $\mathfrak{b} \subset \mathfrak{sl}(1, 1)$. It has trivial bracket, and $H^\bullet(\mathfrak{b}, k) = k[x] \otimes \Lambda[t]$, where x is the dual of $e_{12} \in \mathfrak{b}$, and t dual of the identity matrix h . The restriction of the standard Lie bialgebra structure of $\mathfrak{gl}(1, 1)$ gives the cobracket $\varphi(e_{12}) = \frac{1}{2}h \wedge e_{12}$. The Poisson bracket on $H^\bullet(\mathfrak{b}, k)$ is defined on the generators as

$$\langle x, x \rangle = \langle t, t \rangle = 0, \quad \langle t, x \rangle = x.$$

Example 1.10 Consider a connected compact Poisson-Lie group G . Denote by $\Omega^\bullet G$ and $P^\bullet G$ the differential forms and polyvector fields respectively. The Poisson structure is given by a bivector $w \in P^2 G$, such that $[w, w] = 0$. This bivector induces the map $p: \Omega^1 G \rightarrow P^1 G$ defined by $p(\alpha)(\beta) = w(\alpha, \beta)$. It can be extended to a morphism of complexes $p: (\Omega^\bullet G, d) \rightarrow (P^\bullet G, d_w = [w, -])$.

Now, let $i_w: \Omega^n G \rightarrow \Omega^{n-2} G$ denote the contraction with bivector w , so that $i_w(\alpha\beta) = w(\alpha, \beta)$. Define degree -1 map B as the commutator $[i_w, d]$. The induced bracket on the forms

$$\langle \omega_1, \omega_2 \rangle = (-1)^{\overline{\omega_1}} (B(\omega_1 \omega_2) - B(\omega_1) \omega_2 - (-1)^{\overline{\omega_1}} \omega_1 B(\omega_2))$$

is mapped to the Schouten bracket of polyvector fields by p . Since i_w provides homotopy for B the induced operator on cohomology and the corresponding bracket vanish.

Recall that in this case the inclusion $\Lambda^\bullet \mathfrak{g}^* \hookrightarrow \Omega^\bullet G$ as left invariant forms is a quasi-isomorphism. The bialgebra structure on \mathfrak{g} is defined by providing a bracket on \mathfrak{g}^* , namely $\varphi^*(x, y) = (di_w(xy))_e$, where x and y are considered both as elements of \mathfrak{g}^* and left invariant forms on G , and the subscript denotes taking value at the identity point e in G .

Even though the map B itself does not preserve left invariant forms, the induced bracket does, and it coincides with the Lie bialgebra cocracket. Indeed, since $B = [i_w, d]$, we have

$$\langle \alpha, \beta \rangle = di_w(\alpha\beta) - i_w d(\alpha\beta) + i_w(d\alpha)\beta - \alpha i_w(d\beta).$$

Expanding the second term using $d(\alpha\beta) = (d\alpha)\beta - \alpha d\beta$, we find

$$\langle \alpha, \beta \rangle = di_w(\alpha\beta) - i_{p(\beta)}(d\alpha) + i_{p(\alpha)}(d\beta).$$

For any Poisson-Lie group $w_e = 0$, therefore $\langle x, y \rangle_e = di_w(xy)_e = \varphi^*(x, y)$. Since the inclusion as left invariant forms induces isomorphism $H^\bullet(\mathfrak{g}, k) \simeq H^\bullet(G, k)$, we see that the 1-shifted Poisson structure on $H^\bullet(\mathfrak{g}, k)$ is trivial.

For the general facts on Poisson-Lie groups used here, we refer to [LW], [Kos].

1.11 Relation to the 2-shifted symplectic structure on BG .

Recall the definition of the 2-shifted symplectic structure on BG from [PTVV]. Let G be an affine smooth algebraic group over field k . Assume that its Lie algebra \mathfrak{g} possesses a non-degenerate invariant symmetric bilinear form. The cotangent complex on the quotient stack BG is $\mathfrak{g}^*[-1]$, so that the bilinear form considered as an element of

$$H_{\text{Hoch}}^0(G, \Lambda^2(\mathfrak{g}^*[-1])) \simeq (S^2 \mathfrak{g}^*)^G,$$

is a 2-shifted 2-form on BG .

Now assume that G is a Drinfeld double of a Poisson Lie group K . Then we have a Manin triple of corresponding Lie algebras $(\mathfrak{g}, \mathfrak{g}_+ = \mathfrak{k}, \mathfrak{g}_- = \mathfrak{k}^*)$, and \mathfrak{g} is equipped with a symmetric bilinear form, such that \mathfrak{k} is identified with an

isotropic subalgebra of \mathfrak{g} . Then the map of the quotient stacks $i: BK \rightarrow BG$ is Lagrangian. It is straightforward to see, using the characterization of Lagrangian maps in [Ca]. The cotangent complex \mathbb{L}_{BK} is identified with $\mathfrak{k}^*[-1] \simeq \mathfrak{g}[-1]$, and $T_{BK} = \mathfrak{k}[1]$ is clearly the kernel of the map $i^*T_{BG} \rightarrow \mathbb{L}_{BK}[2]$, induced by the bilinear form.

Here we work with the profinite completion \widehat{G} of G at identity. The ring of functions on \widehat{G} is $(U\mathfrak{g})^*$, and for the quotient $B\widehat{G}$ the Hochschild cochain complex $C_{\text{Hoch}}^\bullet(U\mathfrak{g}, k)$ is quasi-isomorphic to the cochain complex of the Lie algebra $C^\bullet(\mathfrak{g}, k)$. The results of this section can be viewed as construction of a 1-shifted Poisson structure on $B\widehat{K}$.

It fits well into the point of view of an n -Poisson structure as a bivector of degree $-n$ on $B\widehat{K}$ ([PTVV]). Since the tangent bundle to \widehat{K} is $\mathfrak{k}[1]$, for $n = 1$ we have

$$H^0(C_{\text{Hoch}}^\bullet(\widehat{K}, S^2(T_{B\widehat{K}}[-n-1]))[n+2]) = H_{\text{Hoch}}^1(\widehat{K}, \Lambda^2\mathfrak{k}) = H^1(\mathfrak{k}, \Lambda^2\mathfrak{k}).$$

Which is a 1-cocycle with coefficients in $\Lambda^2\mathfrak{k}$. The co-Jacobi condition for the cocycle is equivalent to the Poisson condition for the bivector.

Because of this analogy we will use the following terminology:

Definition 1.12 *The infinitesimal quotient of an affine dg-scheme $X = \text{Spec } A$ by an action of a Lie algebra \mathfrak{g} is the spectrum of dg-algebra (in general with components of both positive and negative degrees) $C^\bullet(\mathfrak{g}, A)$.*

1.13 Formal classifying space of a graded Lie algebra.

In the case when \mathfrak{g} is a graded Lie algebra we define the *formal classifying space* $B\mathfrak{g}$ of \mathfrak{g} as affine dg-scheme $\text{Spec } C^\bullet(\mathfrak{g}, k)$. The tangent bundle to $B\mathfrak{g}$ is $\mathfrak{g}[1]$ with adjoint action of \mathfrak{g} , so that the complex of derivations $\text{Der}(C^\bullet(\mathfrak{g}))$ is isomorphic to $C^\bullet(\mathfrak{g}, \mathfrak{g}[1])$.

Now, if \mathfrak{g} is equipped with an n -shifted symmetric invariant non-degenerate scalar product $(-, -)$, then $B\mathfrak{g}$ is an $(n+2)$ -shifted symplectic manifold with the form

$$\omega \in H^{n+2}(\mathfrak{g}, \Lambda^2(\mathfrak{g}^*[-1])) = H^0(\mathfrak{g}, (S^2\mathfrak{g}^*)[n]),$$

corresponding to the scalar product. Let us describe the Poisson bracket for this symplectic structure.

As before, for an element a of \mathfrak{g} we denote by \overline{a} its degree in \mathfrak{g} . Let $p: \mathfrak{g} \rightarrow \mathfrak{g}^*[n]$ be the isomorphism induced by the scalar product, then the degree of $p(a)$ as an element in $C^\bullet(\mathfrak{g})$ is $|p(a)| = \overline{a} + n + 1$. On the linear functions the $(n+2)$ -shifted Poisson bracket is given by

$$\{p(a), p(b)\} = (-1)^{\overline{a}}(a, b).$$

The Jacobi identity is trivially satisfied. To establish the compatibility with the differential:

$$\{dp(a), p(b)\} + (-1)^{|p(a)|-n}\{p(a), dp(b)\} = d\{p(a), p(b)\} = 0,$$

we observe that $\{p(a), dp(b)\} = Ad_a(p(b))$, then use \mathfrak{g} -invariance of isomorphism p and commutation of the Lie bracket of \mathfrak{g} .

Now let $(\mathfrak{g}, \mathfrak{g}_+ = \mathfrak{k}, \mathfrak{g}_- = \mathfrak{k}^*[n])$ be an n -shifted Manin triple. Denote by $i: B\mathfrak{k} \rightarrow B\mathfrak{g}$ the map of the formal classifying spaces induced by the inclusion of \mathfrak{k} as a subalgebra of \mathfrak{g} . We have the short exact sequence of \mathfrak{k} -modules $\mathfrak{k}[1] \rightarrow \mathfrak{g}[1] \rightarrow \mathfrak{g}_-[1] \simeq \mathfrak{k}^*[n+1]$. It identifies the tangent bundle $T_{B\mathfrak{k}} = \mathfrak{k}[1]$ with the kernel of the map $i^*T_{B\mathfrak{g}} \rightarrow L_{B\mathfrak{k}}[n+2]$. Therefore the map $i: B\mathfrak{k} \rightarrow B\mathfrak{g}$ is Lagrangian ([Ca]).

2 Poisson structures on the derived varieties of complexes.

Let V^\bullet be a graded vector space over a field k of characteristic 0, with only finitely many non-zero spaces, and each V^i of finite dimension. Recall construction of $R\text{Com}(V^\bullet)$ from [KP]. Consider the graded Lie algebra $\text{End}(V)$ of endomorphisms of the total vector space or all degrees. Denote by \mathfrak{q} the parabolic subalgebra

$$\mathfrak{q} = \bigoplus_{i \leq j} \text{Hom}(V^i, V^j)[i-j].$$

We have the Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$, where \mathfrak{l} is the Levi subalgebra and \mathfrak{n} the nilpotent radical of \mathfrak{q} :

$$\mathfrak{l} = \mathfrak{gl}(V^\bullet) = \bigoplus_i \mathfrak{gl}(V^i), \quad \mathfrak{n} = \bigoplus_{i > j} \text{Hom}(V^i, V^j)[i-j].$$

Definition 2.1 *The derived variety of complexes $R\text{Com}(V^\bullet)$ is an affine dg-scheme $\text{Spec } C^\bullet(\mathfrak{n}, k)$.*

Choose a basis $\{v_i\}$ of V^\bullet , and set $\alpha_i = j$ if the corresponding basis vector $v_i \in V^j$. For convenience enumerate basis vectors so that $\alpha_i \leq \alpha_{i'}$ if $i < i'$. Denote by $e_{ij} = v_i \otimes v_j^*$, the basis vector of $\text{End}(V)$ of degree $\overline{e_{ij}} = \alpha_j - \alpha_i$.

2.2 Poisson bracket induced by the standard Lie bialgebra structure.

We may identify \mathfrak{q} with the Lie subalgebra of block upper triangular matrices (including the diagonal blocks) and \mathfrak{n} with the block strict upper triangular matrices. Let \mathfrak{b}_+ be the Borel subalgebra of upper triangular matrices, then using the same construction as in example 1.8, we obtain a Lie bialgebra structure on \mathfrak{q} .

Now let $\mathfrak{l}_1 = \mathfrak{l} \cap \mathfrak{sl}(V)$ and $\mathfrak{q}_1 = \mathfrak{q} \cap \mathfrak{sl}(V)$, where $\mathfrak{sl}(V)$ is the subalgebra of $\text{End}(V)$ consisting of elements with supertrace 0. The orthogonal of \mathfrak{q}_1 , identified with $\mathfrak{n}_- \oplus k$, is an ideal in $\text{End}(V)^*$ with respect to the Lie cobracket, hence the bialgebra structure restricts to \mathfrak{q}_1 as well.

By proposition 1.5 we have a 1-shifted differential Poisson structures on the $C^\bullet(\mathfrak{q}, k)$ and $C^\bullet(\mathfrak{q}_1, k)$. First we will show that Poisson bracket on $C^\bullet(\mathfrak{q})$ induces trivial bracket on the cohomology. Consider the Hochschild-Serre spectral sequence associated to the quotient $\mathfrak{q}/\mathfrak{n}$:

$$E_2^{pq} = H^p(\mathfrak{l}, H^q(\mathfrak{n}, k)) \Rightarrow H^{p+q}(\mathfrak{q}, k).$$

Since \mathfrak{l} is a reductive Lie algebra, it is sufficient to look at submodule of \mathfrak{l} -invariants in $H^\bullet(\mathfrak{n}, k)$. Because of weight considerations (\mathfrak{l} contains the Cartan subalgebra) $H^\bullet(\mathfrak{n})^\mathfrak{l} = H^0(\mathfrak{n}) = k$, so that the spectral sequence degenerates and $H^\bullet(\mathfrak{q}, k) = H^\bullet(\mathfrak{l}, k)$.

After changing base from k to \mathbb{C} , it remains to show that the Poisson bracket on $H^\bullet(\mathfrak{gl}_n(\mathbb{C}))$ is trivial. This can be done using the argument similar to example 1.10.

However the Poisson bracket induced on $H^\bullet(\mathfrak{q}_1, k)$ in general is not trivial, as was seen in example 1.9.

Theorem 2.3 *The infinitesimal quotient (in the sense of 1.12) of the derived variety of complexes $\mathrm{RCom}(V^\bullet)/\mathfrak{l}_1$ has structure of a 1-shifted Poisson manifold.*

Follows immediately from the discussion above and observation that

$$\mathrm{RCom}(V^\bullet)/\mathfrak{l}_1 := \mathrm{Spec} C^\bullet(\mathfrak{l}_1, C^\bullet(\mathfrak{n}, k)) \simeq \mathrm{Spec} C^\bullet(\mathfrak{q}_1, k).$$

Remark 2.4 Instead of taking the infinitesimal quotient we could consider the quotient stack $\mathrm{RCom}(V^\bullet)/GL(V^\bullet)_1$, where $GL(V^\bullet) = \prod_i GL(V^i)$, and $GL(V^\bullet)_1$ is the subgroup consisting of elements (g_1, \dots, g_n) with

$$\prod_i (\det g_i)^{(-1)^i} = 1.$$

One may ask if the 1-Poisson structure of the theorem lifts to a Poisson bivector on this quotient stack.

2.5 $\mathfrak{gl}(n, n)$ as a Drinfeld double and Poisson structures on the derived variety of complexes.

Let $\mathfrak{p} = \mathfrak{gl}(n, n)$, fix a Borel subalgebra \mathfrak{b}_+ in \mathfrak{p} and define bilinear form $(-, -)$ as in 1.8. Fix also a permutation θ of the set $\{1, 2, \dots, n\}$, and denote by \mathfrak{h}_+^θ and \mathfrak{h}_-^θ the subspaces of Cartan subalgebra \mathfrak{h} of \mathfrak{p} spanned by $\{a_i + a_{n+\theta(i)}\}$, and $\{a_i - a_{n+\theta(i)}\}$ respectively, $1 \leq i \leq n$. Let $\mathfrak{g}^\theta = \mathfrak{p}_+^\theta = \mathfrak{n}_+ \oplus \mathfrak{h}_+^\theta$ and $\mathfrak{p}_-^\theta = \mathfrak{n}_- \oplus \mathfrak{h}_-^\theta$.

It is immediate to see that \mathfrak{p}_+^θ and \mathfrak{p}_-^θ are isotropic subalgebras of \mathfrak{p} , and the bilinear form identifies \mathfrak{p}_-^θ with the dual of \mathfrak{p}_+^θ . Therefore we have a (non-shifted) Manin triple $(\mathfrak{p}, \mathfrak{p}_+^\theta, \mathfrak{p}_-^\theta)$, providing \mathfrak{g}^θ with a Lie bialgebra structure.

Notice that this is not a restriction of the standard Lie bialgebra structure on $\mathfrak{gl}(n, n)$ to \mathfrak{g}^θ , except when $n = 1$, because in general the standard Lie cobracket doesn't restrict to \mathfrak{g}^θ .

2.6 We may relate the cohomology with trivial coefficients of \mathfrak{g}^θ to the cohomology of the nilpotent radical \mathfrak{n}_+ computed in [KP]. First, let us restrict to the quasi-isomorphic subcomplex $C^\bullet(\mathfrak{g}^\theta)^{\mathfrak{h}_+^\theta}$ of \mathfrak{h}_+^θ -invariant elements. Now, since

$$C^\bullet(\mathfrak{g}^\theta)^{\mathfrak{h}_+^\theta} = \Lambda^\bullet(\mathfrak{h}_+^\theta)^* \otimes C^\bullet(\mathfrak{n}_+)^{\mathfrak{h}_+^\theta},$$

with the differential induced by that of the cochain complex of \mathfrak{n}_+ and zero differential in the exterior powers, we find

$$H^\bullet(\mathfrak{g}^\theta) = \Lambda^\bullet(\mathfrak{h}_+^\theta)^* \otimes H^\bullet(\mathfrak{n}_+)^{\mathfrak{h}_+^\theta}.$$

The \mathfrak{h}_+^θ -invariant subspace of $H^\bullet(\mathfrak{n}_+)$ is the subspace with weights belonging to the configuration sector $\mathfrak{S}(\theta, \mathfrak{b}_+)$ (see *op. cit.* 2.3).

2.7 Let V^\bullet be a graded vector space with each V^i of dimension at most 1, such that both total odd and total even dimensions are n . Then after appropriate shuffle permutation $\text{End}(V)$ as $\mathbb{Z}/2$ -graded Lie algebra is isomorphic to $\mathfrak{gl}(n, n)$, and ideal \mathfrak{n} defined at the beginning of this section corresponds to the nilpotent radical of some Borel subalgebra \mathfrak{b}_+ . In this setting we have

Proposition 2.8 *For each permutation $\theta \in S_n$, the infinitesimal quotient of $\text{RCom}(V^\bullet)$ by the action of \mathfrak{h}_+^θ has 1-shifted Poisson structure. Each such quotient is a Lagrangian in the formal classifying space (see 1.13):*

$$B\mathfrak{gl}(V) := \text{Spec } C^\bullet(\mathfrak{gl}(V), k).$$

Moreover, in section 3 we will see that these quotients are up to homotopy BV-manifolds.

2.9 Derived variety of 1-periodic complexes.

As another application of the machinery developed here we would like to give a related example of non-shifted Poisson structure on the derived variety of 1-periodic complexes.

Let V^\bullet be a graded vector space with W in each degree $i \in \mathbb{Z}$. Define the space $\text{PCom}(W)$ of 1-periodic complexes on V^\bullet as a subspace of $\text{End}(W)$ consisting of all square-zero elements d , so that

$$\dots \xrightarrow{d} W \xrightarrow{d} W \xrightarrow{d} W \xrightarrow{d} \dots$$

is an unbounded complex. Let $\{e_{ij}\}$ be a basis of $\text{End}(W)$, and $\{x_{ij}\}$ the corresponding dual basis of $\text{End}(W)^*$. The Lie bracket on this dual space provides the Kirillov-Kostant Poisson bracket on $\text{End}(W)$.

$\text{PCom}(W)$ is an affine variety, its ring of functions is the quotient $S^\bullet(\text{End}(W)^*)/I$, where I is the ideal generated by $\sum_k x_{ik}x_{kj}$, for all i, j . It is straightforward to check that the Kirillov-Kostant bracket induces a well-defined Poisson structure on $\text{PCom}(W)$. The corresponding symplectic foliation is the decomposition of $\text{PCom}(W)$ into the union of coadjoint orbits.

Before we define the derived version of $\text{PCom}(W)$ consider the next example.

Example 2.10 Let A be a finite dimensional symmetric Frobenius algebra (in degree 0), i.e., an associative algebra equipped with an isomorphism of A -bimodules $\theta: A \rightarrow \text{Hom}(A, k)$. This is equivalent to having a non-degenerate symmetric bilinear form on A , satisfying the invariance condition $(ab, c)_A = (a, bc)_A$.

Consider the graded Lie algebra $\tilde{A} = tA[t]$, with $\deg t = 1$, so that the Lie bracket is defined as $[at^m, bt^n] = (ab - (-1)^{mn}ba)t^{m+n}$. We want to show that \tilde{A} has a (-1) -shifted Lie bialgebra structure. In order to do this consider the triple $(A((t^{-1})), \tilde{A}, A[[t^{-1}]])$, with the bilinear form $(f, g) = \text{res}_\infty(f, g)_A$, i.e., the coefficient of t^1 in the series $(f, g)_A$. This is clearly a non-degenerate bilinear form on $A((t^{-1}))$, it is symmetric since $(-, -)_A$ is symmetric and the form is non-trivial only if one of the homogenous terms is of even degree.

It remains to show invariance. Let m, n, l are such that $m + n + l = 1$, then using symmetry and invariance of $(-, -)_A$ and the fact that $(-1)^{mn} = (-1)^{(n+l+1)n} = (-1)^{nl}$ we have

$$\begin{aligned} ([at^m, bt^n], ct^l) &= (ab - (-1)^{mn}ba, c)_A t = \\ &= ((a, bc)_A - (-1)^{mn}(ba, c)_A)t = (a, bc - (-1)^{nl}cb)_A t = (at^m, [bt^n, ct^l]). \end{aligned}$$

The Lie subalgebras \tilde{A} and $A[[t^{-1}]]$ are isotropic with respect to this form, so we have a (-1) -shifted Manin triple. By proposition 1.5 the cochain complex $C^\bullet(\tilde{A}, k)$ is a differential Poisson algebra.

Now return to the space of 1-periodic complexes. Let $A = \text{End}(W)$, with the trace scalar product $(a, b) = \text{tr}(ab)$, for any $a, b \in A$. Construct Lie algebras \tilde{A} and $A[[t^{-1}]]$ as in example 2.10.

Definition 2.11 *The derived variety of 1-periodic complexes on a vector space W is an infinite dimensional affine dg-scheme*

$$\text{RCom}(W) = \text{Spec } C^\bullet(\tilde{A}, k) = \text{Spec } S^\bullet(A[[t^{-1}]])$$

In particular it is easy to see that the classical variety $\text{PCom}(W) = \text{Spec } H^0(\tilde{A}, k)$.

From the discussion of example 2.10 it immediately follows that

Theorem 2.12 *The derived variety of 1-periodic complexes $\text{RCom}(W)$ is a Poisson dg-manifold, and the restriction of the Poisson structure to $H^0(\tilde{A}, k)$ coincides with the standard Kirillov-Kostant Poisson structure described above.*

3 Batalin-Vilkovisky algebra structure.

A BV_∞ -algebra A (cf. [CL]) is a commutative graded algebra equipped with an operator $D: A \rightarrow A[[h]]$ of degree 1, such that $D^2 = 0$, $D(1) = 0$ and in the

expansion

$$D = \sum_{i=1}^{\infty} D_i h^{i-1},$$

each D_i is a differential operator of order $\leq i$.

A differential BV algebra is a BV_{∞} -algebra with all $D_i = 0$ for $i \geq 3$. In this case we take h to be of degree 2, and write $D = d + Bh$, so that d is a differential in A of degree +1 and B is an operator of order ≤ 2 and degree -1 . It will be convenient for us to unpack this definition.

Definition 3.1 *A differential Batalin-Vilkovisky algebra is a mixed complex (C^{\bullet}, d, B) (in the sense of [L]), such that (C^{\bullet}, d) is a differential graded algebra and (C^{\bullet}, B) is a BV-algebra.*

3.2 Let \mathfrak{g} be a non-shifted graded Lie bialgebra, then as we saw before the cochain complex $C^{\bullet}(\mathfrak{g}, k) = S^{\bullet}(\mathfrak{g}^*[-1])$ has a structure of Gerstenhaber algebra. As a graded vector space it can be identified with the chain complex of the dual Lie algebra $C_{\bullet}(\mathfrak{g}^*, k)$. The chain differential B considered as a degree -1 operator on the cochain complex is a second order differential operator, with $B^2 = 0$ and $B(1) = 0$. It equips $C^{\bullet}(\mathfrak{g})$ with a BV algebra structure, generating the Gerstenhaber bracket constructed before: for any $x, y \in C^{\bullet}(\mathfrak{g})$ of degrees $|x|$ and $|y|$

$$\langle x, y \rangle = (-1)^{|x|}(B(xy) - (Bx)y - (-1)^{|x|}xB y).$$

Notice that in general it is not a differential BV-algebra, since operators B and d do not commute. However, we will describe a class of Lie bialgebras such that $C^{\bullet}(\mathfrak{g}, k)$ is a differential BV-algebra at least up to a quasi-isomorphism.

Since B is a second order operator and d is a first order operator, a priori the commutator $[B, d]$ is a second order operator. However the cocycle condition for the Lie bialgebra ensures that it is in fact a derivation.

Lemma 3.3 *Let $(C^{\bullet}, d, \langle, \rangle)$ be a differential Gerstenhaber algebra, with the bracket generated by a second order operator B . Then the commutator $\Delta = Bd + dB: C^{\bullet} \rightarrow C^{\bullet}$ is a derivation, i.e. it satisfies the Leibniz identity: $\Delta(xy) = \Delta(x)y + x\Delta(y)$.*

Since C^{\bullet} is a differential Gerstenhaber algebra, we have $d\langle x, y \rangle = \langle dx, y \rangle + (-1)^{|x|+1}\langle x, dy \rangle$. Rewriting brackets using codifferential B , on the left hand side we have

$$(-1)^{|x|}dB(xy) + (-1)^{|x|+1}dBx \cdot y + Bx \cdot dy - dx \cdot By + (-1)^{|x|+1}x \cdot dBy,$$

and on the right hand side:

$$\begin{aligned} & (-1)^{|x|+1}B(dx \cdot y) + (-1)^{|x|}Bdx \cdot y - dx \cdot By - \\ & B(x \cdot dy) + Bx \cdot dy + (-1)^{|x|}x \cdot Bdy. \end{aligned}$$

Comparing these two expressions we establish the required identity.

Corollary 3.4 *Let $(C^\bullet, d, \langle, \rangle, B)$ be as in the lemma, $\Delta = [B, d]$. The kernel $\text{Ker } \Delta$ is a differential BV-algebra, and the inclusion $\text{Ker } \Delta \hookrightarrow C^\bullet$ is a map of differential Gerstenhaber algebras.*

Commutation of Δ with B and d implies that they restrict to the $\text{Ker } \Delta$. Since Δ is a derivation, $\text{Ker } \Delta$ is also a subalgebra. Finally the bracket, being generated by B , can also be restricted to the kernel.

Corollary 3.5 *Let (C^\bullet, d, B) be a differential BV-algebra, with bracket $\langle -, - \rangle$ induced by B . Then $(C^\bullet, d, \langle -, - \rangle)$ is a differential Gerstenhaber algebra.*

In a mixed complex the commutator $\Delta = 0$, so that the relation in the previous lemma is trivially satisfied. This in turn equivalent to d being a derivation of the Gerstenhaber bracket.

Corollary 3.6 *If \mathfrak{g} is a graded Lie bialgebra, then the commutator $\Delta = [B, d]$ is a derivation.*

Corollary 3.7 *Let \mathfrak{g} be an involutive graded Lie bialgebra, i.e., the composition*

$$\mathfrak{g} \xrightarrow{\varphi} \Lambda^2 \mathfrak{g} \xrightarrow{[\cdot, \cdot]} \mathfrak{g}$$

is trivial. Then $C^\bullet(\mathfrak{g})$, with d and B as before, is a differential BV-algebra.

The vanishing of the composition of Lie bracket and cobracket implies $\Delta(x) = 0$ for all $x \in C^1(\mathfrak{g})$. Since $C^\bullet(\mathfrak{g})$ is generated as an algebra by $C^1(\mathfrak{g})$ the statement follows.

Example 3.8 Let \mathfrak{g} be a coboundary Lie bialgebra, i.e., with the cobracket given by $\varphi(x) = [r, x]$ for some $r \in S^2(\mathfrak{g}[1])$, with $[r, r] \in (S^3(\mathfrak{g}[1]))^{\mathfrak{g}}$. Here $[-, -]$ denotes the Schouten bracket, generated by the differential δ of the chain complex. We want to find a condition for r for the cochain complex $C^\bullet(\mathfrak{g})$ to be a mixed complex. It will be more convenient to look at the dual to the commutator:

$$\Delta^*(x) = [\delta, \varphi](x) = \delta[r, x] + [r, \delta x] = [\delta r, x].$$

Therefore \mathfrak{g} is an involutive Lie bialgebra, i.e., $\Delta = 0$ if and only if δr lies in the center of \mathfrak{g} . For example, this is the case for $\mathfrak{sl}(1, 1)$ with the standard Lie bialgebra structure, where $r = \frac{1}{2}e_{12} \wedge e_{21}$, and δr is the identity matrix.

Example 3.9 Now look at the graded Lie algebra $\mathfrak{g} = \mathfrak{gl}(m, n)$ with the standard Lie bialgebra structure. It is a coboundary Lie bialgebra with

$$r = \frac{1}{2} \sum_{i < j} e_{ij} \wedge e_{ji}.$$

Although in this case δr doesn't in general belong to the center of \mathfrak{g} , it lies in the Cartan subalgebra of \mathfrak{g} . In the case of $\mathfrak{gl}(n)$ it's the half-sum of the positive roots: $\delta r = \frac{1}{2} \text{diag}(n-1, n-3, \dots, -n+1)$, and for $\mathfrak{gl}(m, n)$, we have $\delta r = \frac{1}{2} \text{diag}(m+n-1, \dots, -m+n+1, m+n-1, \dots, m-n+1)$. Therefore $\Delta^* = [\delta r, -]$ acts semi-simply on the chain complex. This is an important case due to the following lemma.

Lemma 3.10 *Let \mathfrak{g} be a graded Lie bialgebra. As before, d and B are the differential and BV operator on the cochain complex $C^\bullet(\mathfrak{g})$. If $\Delta = [B, d]$ acts semi-simply on \mathfrak{g}^* , then the embedding $\text{Ker } \Delta \hookrightarrow C^\bullet(\mathfrak{g})$ is a quasi-isomorphism of differential Gerstenhaber algebras.*

Using the Leibniz identity of lemma 3.3 we see that Δ acts semi-simply on all $C^n(\mathfrak{g})$. Since it commutes with the differential we have the direct sum decomposition of the cochain complex $C^\bullet(\mathfrak{g}) = \bigoplus_\lambda C^\bullet(\mathfrak{g})^\lambda$ into eigenspaces of Δ . Now, by definition, B is a homotopy for Δ , so that Δ acts trivially on the cohomology groups of \mathfrak{g} , therefore $\text{Ker } \Delta = C^\bullet(\mathfrak{g})^{\lambda=0}$ is a quasi-isomorphic subcomplex of $C^\bullet(\mathfrak{g})$.

Example 3.11 Let us return back to the setting of section 2. The quotient of the derived variety of complexes $\text{RCom}(V)$ on a graded vector space V^\bullet by the Lie algebra \mathfrak{l}_1 of infinitesimal automorphisms with supertrace 0, is identified with the spectrum of the differential graded algebra $S^\bullet(\mathfrak{q}_1^*[-1])$ where \mathfrak{q}_1 is the parabolic subalgebra of endomorphisms of the total space V with 0 supertrace and of non-negative degree. The r-matrix for the standard Lie bialgebra is given by the same expression as in the example 3.9. We express r as an element in $\Lambda^2 \mathfrak{g}$ rather than $S^2(\mathfrak{g}[1])$ to simplify signs.

Because δr is a diagonal matrix, it acts semisimply on $S^\bullet(\mathfrak{q}_1^*[-1])$, and according to the lemma the kernel of the coadjoint action $Ad_{\delta r}^*$ is a quasi-isomorphic Gerstenhaber subalgebra. In other words

Theorem 3.12 *The infinitesimal quotient of the derived variety of complexes $\text{RCom}(V^\bullet)/\mathfrak{l}_1$ is a homotopy BV-manifold.*

Example 3.13 It is clear from the construction of the Lie bialgebra structure on \mathfrak{g}^θ from section 2.5 that the action of the commutator Δ on $(\mathfrak{g}^\theta)^*$ is semi-simple, and according to lemma 3.10, $\text{Ker } \Delta$ is a quasi-isomorphic Gerstenhaber subalgebra of $C^\bullet(\mathfrak{g}^\theta)$.

Proposition 3.14 *Under assumptions of example 2.7 the infinitesimal quotient $\text{RCom}(V^\bullet)/\mathfrak{h}_+^\theta$ is a homotopy BV-manifold.*

References

- [Ca] D. Calaque. Lagrangian structures on mapping stacks and semi-classical TFTs. [arxiv:math.AT/1306.3235](#).
- [CL] K. Cieliebak and J. Latschev. The role of string topology in symplectic field theory. [arxiv:math.SG/0706.3284](#).
- [CP] V. Chari, A. Pressley. Quantum Groups. Cambridge; New York, N.Y.: Cambridge University Press, 1994.
- [CS] M. Chas, D. Sullivan. String Topology. [arxiv:math.GT/9911159](#).
- [Kow] Niels Kowalzig. Batalin-Vilkovisky algebra structures on (Co)Tor and Poisson bialgebroids.
- [Kos] J.-L. Koszul. Crochet de Schouten-Nijenhuis et cohomologie. Astérisque 1985. Numero Hors Serie, 257-271. The mathematical heritage of Élie Cartan (Lyon, 1984).
- [KP] M. Kapranov, S. Pimenov. Derived varieties of complexes and Kostant’s theorem for $\mathfrak{gl}(m|n)$. [arxiv:math.AG/1504.00339](#)
- [L] Jean-Louis Loday. Cyclic Homology. Springer-Verlag, 1998.
- [LW] J. H. Lu, A. Weinstein. Poisson-Lie Groups, Dressing Transformations and Bruhat Decompositions. *J. Diff. Geom.* **31**, 501–526.
- [Mel] V. Melani. Poisson bivectors and Poisson brackets on affine derived stacks. [arxiv:math.AG/1409.1863](#).
- [Men] Luc Menichi. Batalin-Vilkovisky algebras and cyclic cohomology of Hopf algebras. [arxiv:math.QA/0311276](#).
- [PTVV] T. Pantev, B. Toën, M. Vaquié, G. Vezzosi. Shifted Symplectic Structures. [arxiv:math.AG/1111.3209](#).
- [V] Izu Vaisman. Lectures on the Geometry of Poisson Manifolds. Progress in mathematics, vol. 118. Basel; Boston: Birkhauser Verlag, 1994.